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Francis Bloch, Nicolas Houy

ECOLE POLYTECHNIQUE
CENTRE NATIONAL DE LA RECHERCHE SCIENTIFIQUE

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Francis BLOCH
Nicolas HOUY

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DEPARTEMENT D'ECONOMIE
Route de Saclay
91128 PALAISEAU CEDEX
(33) 1 69333033
http://www.enseignement.polytechnique.fr/economie/
mailto:chantal.poujouly@polytechnique.edu

Optimal Assignment of Durable Objects to Successive Agents*

Francis Bloch[†] Nicolas Houy [‡]

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Abstract

This paper analyzes the assignment of durable objects to successive generations of agents who live for two periods. The optimal assignment rule is stationary, favors old agents and is determined by a selectivity function which satisfies an iterative functional differential equation. More patient social planners are more selective, as are social planners facing distributions of types with higher probabilities for higher types. The paper also characterizes optimal assignment rules when monetary transfers are allowed and agents face a recovery cost, when agents' types are private information and when agents can invest to improve their types.

JEL Classification Numbers: C78, D73, M51

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†Department of Economics, Ecole Polytechnique, 91128 Palaiseau France, tel: +33 1 69 33 30 45, francis.bloch@polytechnique.edu

‡Department of Economics, Ecole Polytechnique, 91128 Palaiseau France, tel: +33 1 69 33 30 15, nhouy@free.fr

1 Introduction

This paper considers durable objects which are successively reassigned to agents. The prime example of objects which are regularly reassigned are positions in organizations and bureaucracies, like executive offices or diplomatic postings. Other examples include offices, rooms in dormitories, computer servers, large scale scientific equipment like telescopes and particle accelerators, hotel rooms and rental cars. In all these examples, agents are given temporary property rights over the durable object for a given period, and cannot be forced to return the object. These temporary property rights create a constraint on the optimal assignment policy chosen by a benevolent social planner, and our objective in this paper is to characterize the efficient dynamic assignment of a durable object to successive generations of players taking into account these individual rationality constraints.

We consider environments where overlapping generations of agents enter the market deterministically, and agents are assigned the object for their entire lifetime. Agents differ in their valuations for the objects, or their qualifications for the positions. Agents characteristics are thus two-dimensional and include a type which determines the value of the assignment, and an age which determines the length of the assignment. Our objective in this paper is to characterize optimal assignment rules in this two-dimensional model, and construct revelation mechanisms when agents' types are privately known.

The basic trade-off between age and value is best understood in a two-period overlapping generations model. When assigning the good to an old or young agent, the social planner makes the following computation. Assigning the good to the old agent for one period has a positive option value, as the good can be reassigned at the end of the period ; assigning the good to the young agent for two periods prevents the planner from reassigning the good immediately. Hence, if the old and good agents were of the same type, it would always be optimal to assign the good to the *old agent*.[1] This line of reasoning shows that, for any type θ of the young agent, the planner will prefer to give to the any old agent of type greater or equal to $\phi(\theta)$, where $\phi(\theta) < \theta$.

The main contribution of the paper is to characterize the selectivity function, $\phi(\theta)$, which is adopted in the optimal assignment policy. This function is defined by an iterative functional differential equation, which is similar to the equations used in physics and mathematics to study dynamical systems with state-dependent delay

[1]The option value of giving a position to an older agent is a well documented historical fact. For example, the history of the papacy records a number of elections where cardinals voluntarily chose the oldest candidate. Oftentimes, these "transition popes" turn out to be the most energetic and effective popes of their times. See Collins (2009) and his account of the reign of two famous "transition popes", John XXII (1316-1334) and John XXIII (1958-1965).

(Eder 1984). While we cannot provide an analytical solution to the functional differential equation, we prove existence and uniqueness of the optimal assignment policy and show that the selectivity function is increasing and concave. We illustrate the optimal assignment policies for uniform and quadratic type distributions, and derive comparative statics properties of the solution. Under some regularity conditions, we show that when the discount factor increases, or when the distribution of types is shifted so that higher types have a higher probability, the social planner becomes more selective, and assigns the object to the young agent less often.

In the second part of the paper, we consider different extensions of the model. First, we analyze the optimal assignment policy when monetary transfers are allowed and agents can be compensated when they return the object. If there is no recovery cost, the optimal assignment policy is identical to the first-best policy without individual rationality constraints: the object is assigned at every period to the agent with the highest value. If an old agent who currently holds the object has a smaller value than the young agent, the young agent can easily transfer money in return for the object and the individual rationality constraint ceases to be binding. With positive recovery costs, the optimal assignment strategy becomes more complex, and involves two different selectivity functions, one which is used at periods where no agent holds the object, and one which is used when the old agent holds property rights over the object and needs to be compensated. We characterize the optimal assignment policies as solutions to systems of differential functional equations, both with fixed and proportional recovery costs. We also illustrate these complex assignment strategies for the uniform distribution.

In a second extension of the model, we analyze direct revelation mechanisms when the types of the agents are privately known. Given the time structure of the assignment rule, we can build different revelation mechanisms. In the first model, we suppose that agents are asked to reveal their types when they enter society as young agents, whether the object is reassigned or not. In the second model, we assume that agents are only asked to reveal their types (and pay a transfer) at periods where the good is reassigned. For both models, we use standard arguments to characterize transfer rules implementing the optimal assignment policy. Not surprisingly, these transfer rules involve a step function, where transfers jump to a flat positive value when the agent's type reaches the threshold value for which the good is assigned to her.

Finally, we investigate the agents' incentives to invest in order to improve their types between the two periods of their lives. We characterize the optimal investment and assignment strategies of the social planner as solutions to a system of simultaneous differential equations. We show that agents' optimal investment is higher when they are sure to retain the object, and illustrate the optimal investment strategy for a uniform distribution of types and a quadratic investment cost function.

Axiomatic characterizations of assignment rules for durable goods have recently been proposed by Kurino (2008) and Bloch and Cantala (2008). Kurino (2008) considers a dynamic extension of Abdulkadiroglu and Sönmez (1999)'s study of house allocation with existing tenants – the first example of an assignment problem with individual rationality constraints –, and analyzes whether the rules proposed in the static paper still satisfy efficiency and incentive compatibility in the dynamic context. Bloch and Cantala (2008) consider a model where agents are assigned to different, vertically related objects, and characterize Markovian assignment rules which satisfy myopic efficiency and fairness. By contrast, in this paper, we consider a simpler model where agents only live two periods and can only be assigned one good. In this simpler model, we are able to characterize dynamically efficient rules, and to discuss the incentive properties of transfer rules.

This paper is also related to the rapidly growing literature on dynamic mechanism design. Parkes and Singh (2003), Athey and Segal (2007), Bergemann and Valimäki (2006) and Gershkov and Moldovanu (2008a, 2008b, 2008c) study dynamic assignment problems, where agents enter sequentially, and participate in a Vickrey-Clarke-Groves revelation mechanism which determines transfers and good allocations. They show that Vickrey-Clarke-Groves mechanisms and optimal stopping rules can be combined to obtain efficient dynamic mechanisms. In these models, objects can only be assigned once at the time of entry. Some of these studies (like Gershkov and Moldovanu (2008b, 2008c)) distinguish between benevolent and revenue-maximizing planners. When agents' types are known, the literature on yield management in management science and operations research (see Talluri and Van Rysin (2004)) provides an in-depth study of optimal pricing strategies.

The rest of the paper is organized as follows. We introduce the model in Section 2. We analyze efficient assignment policies in Section 3. Section 4 is devoted to the three extensions of the model and Section 5 concludes.

2 The Model

2.1 Agents

We consider the assignment of a single durable object to overlapping generations of agents. Time is discrete and runs as $t = 1, 2, ...\infty$. At each period t, one new agent enters society. Agents live for two periods, so that at each period, society consists exactly of one young and one old agent. All agents share the same discount factor $\delta \in (0, 1)$.

Agents are characterized by their type θ which measures the flow of utility generated by the assignment of the object. We suppose that types are drawn before an agent enters society and last for the agents' entire lifetime. Types of successive agents are drawn independently from the same distribution F, which is assumed to be non-atomic, have full support on a compact interval $\Theta = [\underline{\theta}, \overline{\theta}] \subset \Re_+$ and admit a continuous density function f. We assume that types are always positive, so that all assignments create positive surplus.[2] At any time t, we denote by θ_t^y and θ_t^o the types of the young and old agents. The model starts at period 0, where the type of the old agent, θ_0^o is given and known to everyone.

The object will be successively assigned to agents present in society. We suppose that agents cannot be forced to relinquish the object. In other words, *we assume that when the object is assigned to a young agent, the young agent keeps it for two periods.* This assumption introduces a strong asymmetry between old and young agents. It implies that, at some periods, the object will not be reassigned. We can thus distinguish between two sets of dates: a set T_0 of periods at which the object is *not reassigned* because the young agent retains it for another period, and a set of dates T_1 at which the object is reassigned. This is illustrated in the following graph:

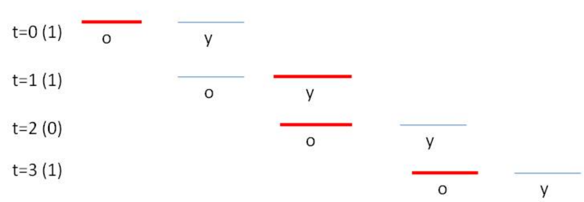


Figure 1: `Overlapping generations and assignments`

In this example, there are three dates at which the object is assigned, dates

[2]This assumption is made without loss of generality. If some types were negative, we could as well consider a distribution where these types form an atom at zero (which would not affect our analysis), as the good will never be assigned to them.

$t = 0, 1, 3$. At date $t = 2$, the object is not assigned. The object is assigned to the old agent at periods $t = 0, 3$ and to the young agent at $t = 1$.

In the baseline model, we suppose that transfers to agents are not allowed, and that the planner only chooses to whom the object is assigned at any date $t \in T_1$. Formally, the planner's control variables are the probabilities p_t^o and p_t^y with which the object is assigned to the old and young agent. The utility flow for a young and old agent at date $t \in T_1$ are given by

$$\begin{aligned} U_t^y &= p_t^y \theta_t^y, \\ U_t^o &= p_t^o \theta_t^o \end{aligned}$$

and at a date $t \in T_0$ by

$$\begin{aligned} U_t^y &= 0, \\ U_t^o &= \theta_t^o. \end{aligned}$$

2.2 Social planner

We assume that the social planner has an infinite horizon and evaluates sequences of payoffs using the same discount factor δ as the agents. At each period $t \in T_1$, the social planner chooses a probability pair (p_t^y, p_t^o). These choices generate a stochastic process over the sequences of assignments $(\theta_0, ..., \theta_t, ...)$ and we assume that the *benevolent* social planner chooses the probability pair in order to maximize the discounted sum of utilities resulting from the assignment:

$$V = \sum_{t=1}^{\infty} \delta^t E[\theta_t],$$

While the social planner can in principle condition his choice $\{(p_t^y, p_t^o),\}$ at period t on the entire history up to period t. A simple application of well-known arguments shows that there is no loss of generality in restricting attention to *Markovian decisions*, where the social planner bases his decision on a state summarizing the payoff-relevant part of the history.[3] In our context, the payoff-relevant part of the history at any date $t \in T_1$ is given by the types of agents present at period t, and we define a state as a vector of types of the young and old agents, (θ^y, θ^o) in $\Theta \times \Theta$.

[3] See Putterman (1994) for an introduction to the literature on Markovian decision problems and the proof that Markovian decision rules are optimal.

3 Optimal assignment of objects

3.1 Characterization of the optimal assignment

We now characterize the optimal assignment by applying Bellman's principle of optimality to the Markovian decision problem of the benevolent social planner. Define the value function at a state (θ^y, θ^o) by

$$\begin{aligned} V(\theta^y, \theta^o) = \max_{p^y, p^o} \quad & p^y(\theta^y(1+\delta) + \delta^2 \int_{\underline{\theta}}^{\overline{\theta}} \int_{\underline{\theta}}^{\overline{\theta}} V(t^y, t^o) f(t^y) f(t^o) dt^y dt^o) \\ & + p^o \theta^o + \delta(1-p^y) \int_{\underline{\theta}}^{\overline{\theta}} V(t^y, \theta^y) f(t^y) dt^y. \end{aligned} \tag{1}$$

In the expression above, we distinguish between the two choices of the social planner: if she allocates the good to the young agent, the young agent keeps it for two periods, and after two periods, the new state will be characterized by two new types which have been drawn according to the distribution F. If, on the other hand, the good is assigned to the old agent, he will keep it only for one period, and in the next period, the type of the old agent will be known, and given by $\theta^o_{t+1} = \theta^y_t$ while the type of the new agent will be drawn according to F. From expression (1), we immediately make one observation. As types are assumed to be positive, the planner's objective is increasing in p^o for p^y fixed, so she will always assign the good to one of the two agents. We can thus replace p^o by $1 - p^y$. Given this,

$$\begin{aligned} V(\theta^y, \theta^o) = \max_{p^y} \quad & p^y(\theta^y(1+\delta) + \delta^2 \int_{\underline{\theta}}^{\overline{\theta}} \int_{\underline{\theta}}^{\overline{\theta}} V(t^y, t^o) f(t^y) f(t^o) dt^y dt^o) \\ & + (1-p^y)(\theta^o + \delta \int_{\underline{\theta}}^{\overline{\theta}} V(t^y, \theta^y) f(t^y) dt^y). \end{aligned}$$

The objective is linear in p^y, so that we obtain a simple characterization of the optimal policy: the planner should either assign the good to the young or the old agent with probability 1, depending on the sign of $\theta^y(1+\delta)+\delta^2 \int_{\underline{\theta}}^{\overline{\theta}} \int_{\underline{\theta}}^{\overline{\theta}} V(t^y, t^o) f(t^y) f(t^o) dt^y dt^o - \theta^o - \delta \int_{\underline{\theta}}^{\overline{\theta}} V(t^y, \theta^y) f(t^y) dt^y$. Next, define the mapping ϕ from Θ to $\Re$ by:

$$\phi(\theta) \equiv \theta(1+\delta) - \delta \int_{\underline{\theta}}^{\overline{\theta}} V(t, \theta) f(t) dt + \delta^2 \int_{\underline{\theta}}^{\overline{\theta}} \int_{\underline{\theta}}^{\overline{\theta}} V(t, z) f(t) f(z) dt dz.$$

We label the function ϕ the *selectivity function* associated to the optimal planner's decision. This terminology is easily understood: the social planner will allocate the good to the old agent if and only if $\theta^0 \geq \phi(\theta^y)$. As the selectivity function fully characterizes the optimal assignment, we now study the properties of this mapping.

Lemma 1 *The function ϕ is strictly increasing, and satisfies $\phi(\theta) < \theta$ for all $\theta \in [\underline{\theta}, \overline{\theta})$, $\phi(\overline{\theta}) = \overline{\theta}$.*

Lemma 1 provides a useful characterization of the optimal assignment for any pair of types (θ^y, θ^o). It shows that optimal assignments are characterized by *threshold rules*, determining for any θ^y, the minimal type of the old agent who is preferred to θ^y, and for any θ^o, the minimal type of the young agent who is preferred to θ^o. Of course, these optimal threshold rules are not independent (one is the inverse of the other), and we focus attention on the threshold rule $\phi(\theta^y)$, which determines the minimal type of the old agent preferred to θ^y. The second part of Lemma 1 captures the "option value" of assigning the object to the old agent. If both agents have the same type, the social planner always prefers to assign the object to the old agent, as she retains the option value of assigning the object to the young agent next period, but can also draw a young agent with higher type. This option value explains the existence of a positive "gap", measured by $\theta - \phi(\theta)$, between the minimal type of the young and old agents. Finally, notice that Lemma 1 holds without any condition on the differentiability of the selectivity function ϕ.

Next, observe that, assuming that ϕ is differentiable, by a simple application of the envelope theorem, we compute:

$$\frac{\partial \int_{\underline{\theta}}^{\overline{\theta}} V(t,\theta) f(t) dt}{\partial \theta} = F(\phi^{-1}(\theta)).$$

Using this observation, we can easily characterize continuously differentiable selectivity functions as solutions to a functional differential equation:

Theorem 1 *There exists an optimal assignment policy characterized by a continuously differentiable selectivity function $\phi(\cdot)$ which is the unique solution of the iterative functional differential equation:*

$$\phi'(\theta) = 1 + \delta - \delta F[\phi^{-1}(\theta)] \tag{2}$$

with initial condition $\phi(\overline{\theta}) = \overline{\theta}$.

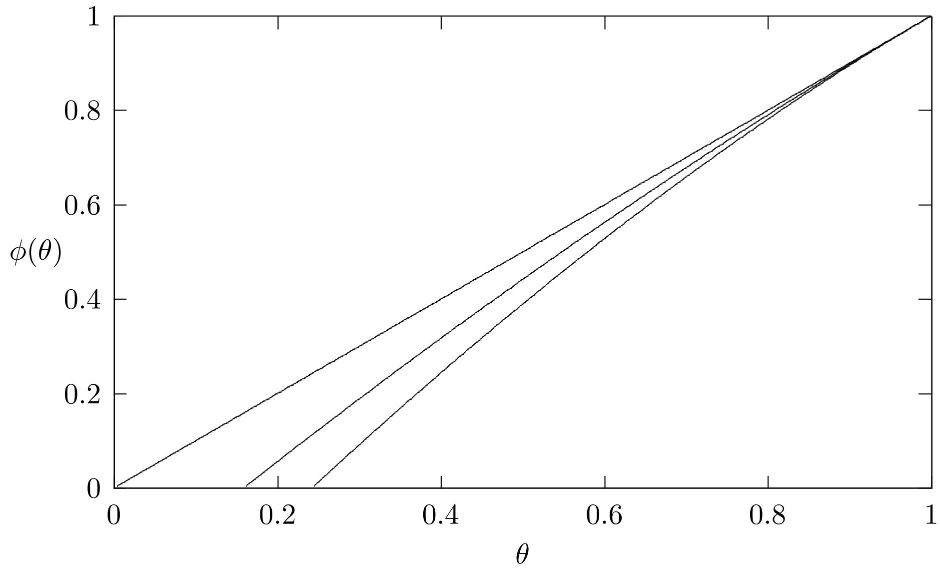
Theorem 1 characterizes the optimal policy as the solution to an iterative functional differential equation. The iterative functional differential equation (2) belongs to a class of differential equations which have been studied in physics and mathematics to analyze dynamical systems with state dependent delays (see Eder (1984), Si and Zhang (2004)). Existence and uniqueness of solutions to this functional equation does not derive from standard theorems on ordinary differential equations, but can nevertheless be obtained using Banach's fixed point theorem in functional spaces. This functional differential equation does not typically admit closed form solutions.[4] Inspection of equation (2) provides additional information on the optimal selectivity function ϕ:

Corollary 1 *The optimal selectivity function $\phi(\cdot)$ is strictly concave for any $\theta \in [\underline{\theta}, \overline{\theta})$, and $\phi'(\overline{\theta}) = 1$.*

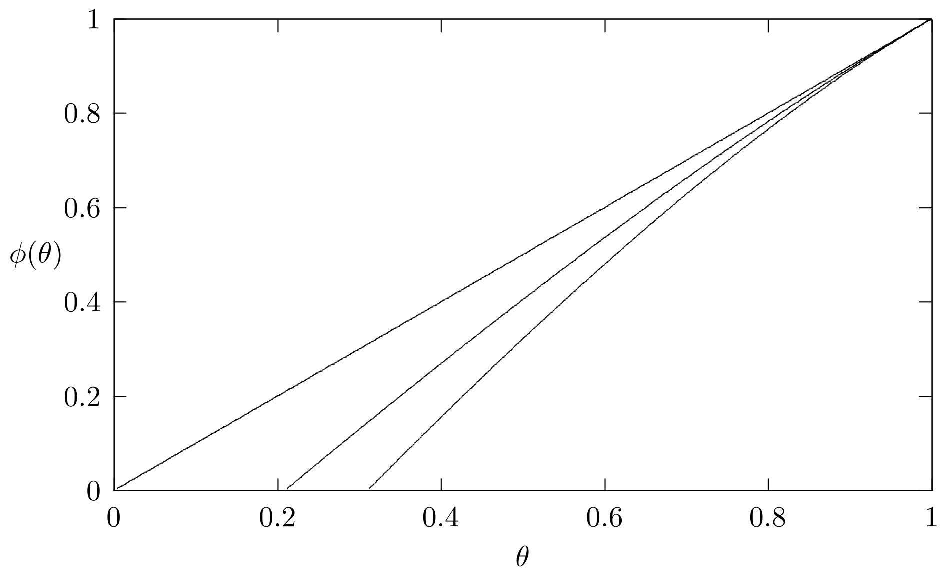
Corollary 1 and Lemma 1 show that even though equation (2) does not admit an analytical solution, the optimal selectivity function possesses remarkable properties. In addition, equation (2) has a simple recursive structure, which stems from the fact that, as $\phi^{-1}(\theta) > \theta$, the value $\phi'(\theta)$ only depends on the value of the function ϕ at points larger than θ. This recursive structure, together with the initial condition $\phi(\overline{\theta}) = \overline{\theta}$, allows for an efficient algorithm to compute numerical solutions to equation (2). Using this algorithm, we compute the optimal selectivity function for the uniform and quadratic distributions over $[0, 1]$.

The first graph shows, for three different values of δ ($\delta = 0, 0.5$ and 1), the optimal selectivity function ϕ for the distribution $F(\theta) = \theta$ over $[0, 1]$. The second graph maps the optimal selectivity function ϕ for the same values of δ and the distribution $F(\theta) = \theta^2$ over $[0, 1]$.

[4] If the distribution F is uniform on $[0, 1]$, the functional differential equation becomes similar to an equation studied by Si and Zhang (2004) who provide one analytical solution to the equation. Unfortunately, the solution they propose does not satisfy the boundary condition $\phi(1) = 1$.



`Figure 2:  Optimal selectivity function (uniform distribution)`



`Figure 3:  Optimal selectivity function (quadratic distribution)`

Figures 2 and 3 illustrate the properties of the selectivity function ϕ, which is increasing, concave, crosses the x-axis at a positive value θ^*, and satisfy $\phi(\overline{\theta}) = \overline{\theta}$ and $\phi'(\overline{\theta}) = 1$ for all distributions and all values of the discount factor. The figures also suggest two comparative statics properties of the selectivity function. From both figures, it appears that, as the discount factor δ increases, the selectivity function ϕ goes down. Furthermore, a comparison of the two figures indicates that the selectivity function ϕ is always lower for the quadratic distribution, which stochastically dominates at the first order the uniform distribution.

3.2 Comparative statics

In this Section, we investigate the two comparative statics properties illustrated by Figures 2 and 3, and discuss the effect of changes in the discount factor and the distribution function on the optimal selectivity function of the social planner.

3.2.1 Changes in the discount factor

In order to analyze the effect of changes in the discount factor δ on the optimal policy, we need to place additional restrictions on the distribution function F. We define the inverse of the hazard rate of the distribution as:

$$\lambda(\theta) = \frac{1 - F(\theta)}{f(\theta)}.$$

We assume that *the hazard rate of the distribution F is monotonic*, so that $\lambda'(\theta) < 0$ for all $\theta \in \Theta$. In addition, we will consider a much stronger assumption on the distribution:

Assumption 1 *The distribution F satisfies the following property: $\lambda'(\theta) + (1 - F(\theta)) \min\{1, f(\theta)^2\} < 0$ for all $\theta \in \Theta$.*

Assumption 1 puts a bound on the density function $f(\theta)$, and typically excludes distributions for which the density exhibits spikes (like a Dirac distribution). Whenever the density is monotonically increasing ($f'(\theta) \geq 0 \; \forall \theta$), the hazard rate satisfies $\lambda'(\theta) \leq 1$, and Assumption 1 is automatically satisfied. In particular, this Assumption holds for the uniform and quadratic distributions illustrated in Figures 2 and 3.

Proposition 1 *For any $\delta' > \delta$, there exists $\epsilon > 0$ such that $\phi(\theta, \delta) > \phi(\theta, \delta')$ for any $\theta \in (\overline{\theta} - \epsilon, \overline{\theta})$. If, in addition, the distribution F has a monotone hazard rate and satisfies Assumption 1, then $\phi(\theta, \delta) > \phi(\theta, \delta')$ for all $\theta \in \Theta$.*

Proposition 1 comprises two separate statements. First, it shows that the optimal policy is monotonic in δ locally around the upper bound of the support of the distribution $\overline{\theta}$. Second, it shows that, under the regularity condition on the distribution F embodied in Assumption 1, the argument can be extended to the entire support of the distribution.

Why do we need a regularity condition to prove that ϕ is decreasing in δ, when a simple intuition seems to suggest that an increase in δ increases the option value of assigning the object to the old agent, resulting in a smaller value of ϕ? A careful inspection of the functional differential equation (2) helps to understand why this simple intuition may be flawed. If $\phi(\theta')$ for all $\theta' > \theta$ is taken to be constant, then an increase in δ immediately implies an increase in the derivative ϕ'. Hence, locally around the upper bound of the distribution, where the value of $\theta' > \theta$ is almost constant, for any $\delta' > \delta$, as $\phi(\overline{\theta}, \delta) = \phi(\overline{\theta}, \delta') = \overline{\theta}$ and $\phi'(\theta, \delta') > \phi'(\theta, \delta)$, we must have $\phi(\theta, \delta') < \phi(\theta, \delta)$. However, for smaller values of θ, this intuition does not extend: any change in δ affects the entire schedule $\phi(\cdot)$ and in particular the value $\phi^{-1}(\theta)$. In order to compute the comparative statics of a change in δ on ϕ, one has to take into account this additional effect in equation (2), which can only be signed under additional regularity restrictions on the distribution. In fact, as the following example illustrates, the optimal policy function may *not be monotonically decreasing in the discount factor* when Assumption 1 is violated.

Example 1 *Let the distribution F over* $[0, 1]$ *be given by:* $F(\theta) = \theta$ *for* $\theta \in (0.995, 1]$ *and* $F(\theta) = 0.001 \times \theta$ *for* $\theta \in [0, 0.995]$.[5] *The following figure displays the difference* $\phi(\theta, 0.999) - \phi(\theta, 0.996)$.

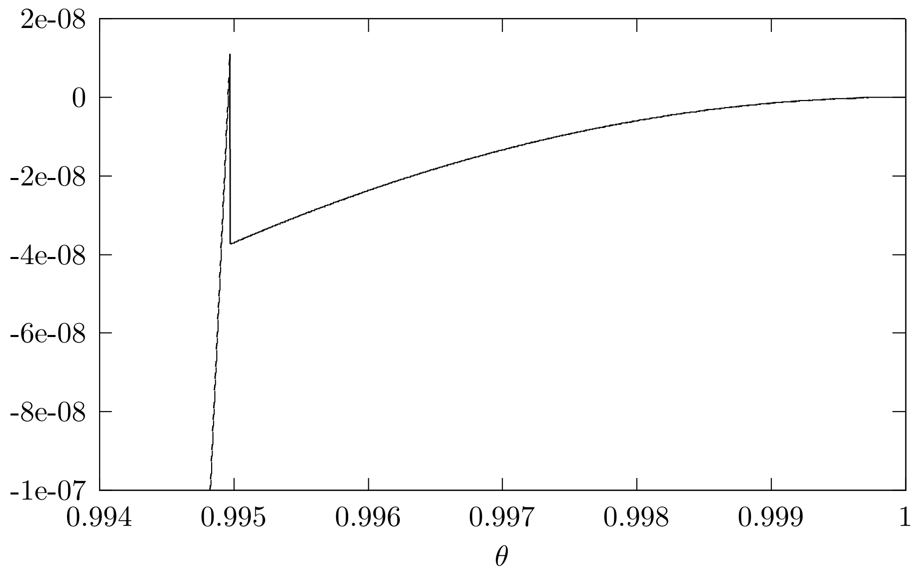


Figure 4: Difference $\phi(\theta, 0.999) - \phi(\theta, 0.996)$

[5]This distribution function is discontinuous, but can be approached by a sequence of continuous distribution functions. Because the problem we are considering is continuous in the distribution F, the lack of monotonicity of this example would also hold for a continuous distribution function.

Figure 4 shows that the selectivity function ϕ is *not monotonic in* δ for the distribution F which spikes at $\theta = 0.0995$. Around the spike, the difference in $\phi(\theta)$ becomes positive, contrary to the statement of Proposition 1.

3.2.2 Changes in the distribution

In order to analyze the effect of changes in the distribution on the optimal policy, we consider a parameterized family of distributions, $F(\theta, \alpha)$ defined over the same support Θ. Distributions with a higher index stochastically dominate at the first order distributions with a lower index : if $\alpha' > \alpha$, $F(\theta, \alpha') < F(\theta, \alpha)$ for all $\theta \in (\underline{\theta}, \overline{\theta})$. Assuming that F is differentiable in both its arguments, we thus suppose that $\frac{\partial F}{\partial \alpha} < 0$ for $\theta \in (\underline{\theta}, \overline{\theta})$, with $\frac{\partial F}{\partial \alpha} = 0$ at $\theta = \underline{\theta}, \overline{\theta}$. This last observation shows that the mapping $\frac{\partial F}{\partial \alpha}$ is not monotonic. However, we will impose a monotonicity restriction by considering the mapping:

$$\mu(\theta, \alpha) = \frac{\frac{\partial F(\theta, \alpha)}{\partial \alpha}}{F(\theta, \alpha)}$$

and assuming that *the mapping* μ *is increasing in* θ. We also define the likelihood ration of the distribution $F(\theta, \alpha)$ as:

$$\nu(\theta, \alpha) = \frac{f(\theta, \alpha)}{F(\theta, \alpha)}.$$

Assumption 2 *The distribution F satisfies the following property: For all α and all θ, $(\ln \nu(\theta, \alpha))' + \nu(\theta, \alpha) \geq (\ln \mu(\theta, \alpha))'$.*

Assumption 2 places a strong regularity condition on the family of distributions $F(\theta, \alpha)$. It is satisfied for the family $F(\theta, \alpha) = \theta^\alpha$ with $\alpha \geq 1$ over $[0, 1]$. For this family, $\mu(\theta, \alpha) = \ln \theta$ and $\nu(\theta) = \frac{\alpha}{\theta}$, and $(\ln \nu(\theta))' + \nu(\theta) = \frac{\alpha - 1}{\theta} \geq \frac{1}{\theta \ln \theta} = (\ln \mu(\theta))'$. Under this Assumption, we establish comparative statics results on the effect of changes in the distribution which parallel the results obtained for changes in δ.

Proposition 2 *For any $\alpha' > \alpha$ there exists $\epsilon > 0$ such that $\phi(\theta, \alpha') < \phi(\theta, \alpha)$ for all $\theta \in (\overline{\theta} - \epsilon, \overline{\theta})$. If in addition the family of distributions $F(\theta, \alpha)$ is such that $\mu(\theta, \alpha)$ is increasing and F satisfies Assumption 2, then $\phi(\theta, \alpha') < \phi(\theta, \alpha)$ for all $\theta \in \Theta$.*

As in the case of changes in the discount factor δ, a simple intuition suggests that an increase in α increases the option value of the old agent, and should result in a lower mapping ϕ. While this intuition is basically correct around the upper end of the support, it ignores the effect of changes in the distribution on the value of $\phi^{-1}(\theta)$ in equation (2). As in the case of the comparative static effects of δ, a strong regularity condition is needed to prove that ϕ is monotonically decreasing in α.

4 Extensions

4.1 Transfers and recovery costs

In the baseline model, we have ruled out monetary transfers between the social planner and agents, and have assumed that the planner cannot recover the object from an agent. If the planner was able to recover the object from an old agent and compensate him with a monetary transfer, she could easily implement the first best allocation, assigning the object to the young agent if and only if $\theta^y \geq \theta^o$. When transfers are allowed, the only constraints which may prevent the social planner from reaching first-best efficiency stem from the costs of recovering the object from an old agent who possesses it. In this section, we consider two specifications for recovery costs: a fixed cost and a proportional cost of recovery. The fixed cost specification corresponds to situations where the cost incurred is independent of the agent's types, as in the case of moving costs or fixed training costs for specific positions. The proportional cost formulation captures situations where the cost is proportional to the type of the old agent, as is the case if the planner must compensate the old agent with a monetary transfer which cannot immediately be extracted from the young agent, and faces a positive cost of public funds.

4.1.1 Fixed recovery costs

We assume that the planner has the possibility of reassigning the good held by the old agent at a cost K. We write down the Markovian decision problem of the benevolent social planner for two situations: $V(\theta^y, \theta^o)$ denotes the value function at a state where the good is not assigned and $W(\theta^y, \theta^o)$ the value function at a state where the good is assigned to the old agent. Straightforward computations show that:

$$
\begin{aligned}
V(\theta^y, \theta^o) = \max_{p^y} \quad & p^y(\theta^y + \delta \int_{\underline{\theta}}^{\overline{\theta}} W(t^y, \theta^y) f(t^y) dt^y) \\
& + (1 - p^y)(\theta^o + \delta \int_{\underline{\theta}}^{\overline{\theta}} V(t^y, \theta^y) f(t^y) dt^y),
\end{aligned}
$$

$$
\begin{aligned}
W(\theta^y, \theta^o) = \max_{q^y} \quad & (1 - q^y)(\theta^o + \delta \int_{\underline{\theta}}^{\overline{\theta}} V(t^y, \theta^y) f(t^y) dt^y) \\
& + q^y(\theta^y - K + \delta \int_{\underline{\theta}}^{\overline{\theta}} W(t^y, \theta^y) f(t^y) dt^y).
\end{aligned}
$$

The preceding expressions are generalizations of the value function obtained in the baseline model. If the social planner allocates the good to the young agent, the young agent keeps it for one period, and the planner is in a state where the good is assigned to the old agent next period. If, on the other hand, the good is assigned to the old agent, he will keep it only for one period and the planner is in a state where the good is not reassigned next period. When the old agent holds the good, the planner must pay the cost K if she wants to assign it to the young agent. As the objectives are linear in p^y and q^y, we obtain a simple characterization of the optimal policy:

- assign the good to the young agent if $\theta^y + \delta \int_{\underline{\theta}}^{\overline{\theta}} W(t^y, \theta^y) f(t^y) dt^y \geq \theta^o + \delta \int_{\underline{\theta}}^{\overline{\theta}} V(t^y, \theta^y) f(t^y) dt^y$ and to the old agent otherwise,
- reassign the good from the old agent to the young if and only if $\theta^y - K + \delta \int_{\underline{\theta}}^{\overline{\theta}} W(t^y, \theta^y) f(t^y) dt^y \geq \theta^o + \delta \int_{\underline{\theta}}^{\overline{\theta}} V(t^y, \theta^y) f(t^y) dt^y$.

These expressions suggest that the optimal policy will again be characterized by *a pair of selectivity functions*, $\phi(\theta)$ and $\gamma(\theta)$, corresponding to situations where the good is not assigned, and when the good is assigned to the old agent. Following the same argument as in the proof of Lemma 1, we can easily show that the two mappings $\phi(\cdot)$ and $\gamma(\cdot)$ are increasing. Replicating the argument of Theorem 1, we also obtain a characterization of the optimal selectivity functions as solutions to a system of iterative functional differential equations.

Proposition 3 *When the planner faces a fixed recovery cost, there exists an optimal assignment policy characterized by a pair of continuously differentiable selectivity function $\phi(\cdot)$ and $\gamma(\cdot)$ which are the unique solutions of the system of iterative functional differential equation:*

$$\phi'(\theta) = 1 + \delta F[\gamma^{-1}(\theta)] - \delta F[\phi^{-1}(\theta)] \tag{3}$$

$$\gamma'(\theta) = 1 + \delta F[\gamma^{-1}(\theta)] - \delta F[\phi^{-1}(\theta)] \tag{4}$$

with initial conditions $\phi(\overline{\theta}) = \overline{\theta}$ and $\gamma(\overline{\theta}) = \overline{\theta} - K$ and the convention that $\gamma^{-1}(\theta) = \overline{\theta}$ for all $\theta \in [\overline{\theta} - K, \overline{\theta}]$.

Proposition 3 shows that our analysis can easily be extended to accommodate situations where the planner faces a recovery cost. When $K = 0$, the two mappings ϕ and γ coincide, and the selectivity function is characterized by the differential equations $\phi(\theta) = \theta$; hence the good is allocated to the agent with the highest type.

When, on the other hand, $\overline{\theta} - \underline{\theta} < K$, we have that $\gamma^{-1}(\theta) = \overline{\theta}$ for all $\theta \in [\underline{\theta}, \overline{\theta}]$, so that the selectivity function is characterized by the unique equation:

$$\phi'(\theta) = 1 + \delta - \delta F(\phi^{-1}(\theta)),$$

as in the baseline model.

Using equations (3) and (4), we easily see that ϕ and γ are both increasing and concave, that $\phi'(\overline{\theta}) = \gamma'(\overline{\theta}) = 1$, that $\phi(\theta) < \theta$ and $\gamma(\theta) < \theta - K$.

The following graphs show, for three different values of δ ($\delta = 0, 0.5$ and 1), the selectivity functions for a uniform distribution and a cost $K = 0.5$. The mapping $\phi(\cdot)$ is in solid lines, and the mapping $\gamma(\cdot)$ in dotted lines.

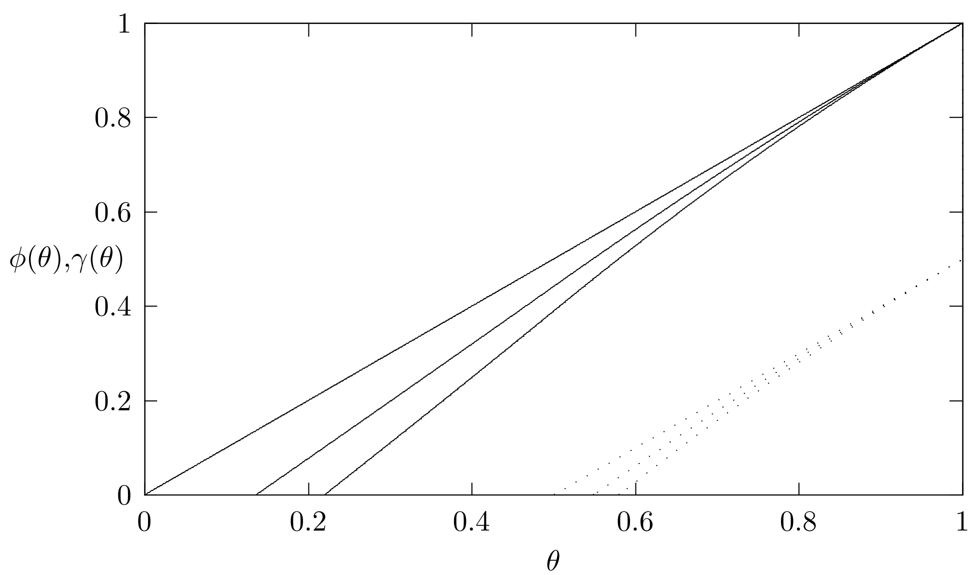


Figure 5: Optimal selectivity function (fixed recovery cost)

4.1.2 Proportional recovery costs

We now suppose that the good can be recovered from the old agent, but at a cost $\kappa\theta^o$ which is proportional to the old agent's surplus. This situation would arise if the planner were to compensate the old agent and support a cost of public funds.[6] With a proportional recovery cost, the values $V(\theta^y, \theta^o)$ and $W(\theta^y, \theta^o)$ at states where the good is not assigned and assigned to the old agent can be computed as:

[6] If the planner can extract a transfer from the young agent to compensate the old, this situation would still arise if the planner must pay the old agent before receiving the young agent's transfer.

$$V(\theta^y,\theta^o) = \max_{p^y} \quad p^y(\theta^y + \delta \int_{\underline{\theta}}^{\overline{\theta}} W(t^y,\theta^y) f(t^y) dt^y) + (1-p^y)(\theta^o + \delta \int_{\underline{\theta}}^{\overline{\theta}} V(t^y,\theta^y) f(t^y) dt^y),$$

$$W(\theta^y,\theta^o) = \max_{q^y} \quad (1-q^y)(\theta^o + \delta \int_{\underline{\theta}}^{\overline{\theta}} V(t^y,\theta^y) f(t^y) dt^y) + q^y(\theta^y - \kappa\theta^o + \delta \int_{\underline{\theta}}^{\overline{\theta}} W(t^y,\theta^y) f(t^y) dt^y).$$

The optimal policy is thus characterized by the following threshold rules:

- assign the good to the young agent if $\theta^y + \delta \int_{\underline{\theta}}^{\overline{\theta}} W(t^y,\theta^y) f(t^y) dt^y \geq \theta^o + \delta \int_{\underline{\theta}}^{\overline{\theta}} V(t^y,\theta^y) f(t^y) dt^y$ and to the old agent otherwise,
- reassign the good from the old agent to the young if and only if $\theta^y + \delta \int_{\underline{\theta}}^{\overline{\theta}} W(t^y,\theta^y) f(t^y) dt^y \geq \theta^o(1+\kappa) + \delta \int_{\underline{\theta}}^{\overline{\theta}} V(t^y,\theta^y) f(t^y) dt^y$.

The optimal selectivity functions are characterized by a system of iterative differential equations:

Proposition 4 *When the planner faces a proportional recovery cost, there exists an optimal assignment policy characterized by a pair of continuously differentiable selectivity function $\phi(\cdot)$ and $\gamma(\cdot)$ which are the unique solutions of the system of iterative functional differential equation:*

$$\phi'(\theta) = 1 + \delta[(1+\kappa)F(\gamma^{-1}(\theta)) - F(\phi^{-1}(\theta)) - \kappa] \quad (5)$$

$$\gamma'(\theta) = \frac{1 + \delta[(1+\kappa)F(\gamma^{-1}(\theta)) - F(\phi^{-1}(\theta)) - \kappa]}{1+\kappa} \quad (6)$$

with initial conditions $\phi(\overline{\theta}) = \overline{\theta}$ and $\gamma(\overline{\theta}) = \frac{\overline{\theta}}{1+\kappa}$ and the convention that $\gamma^{-1}(\theta) = \overline{\theta}$ for all $\theta \in [\frac{\overline{\theta}}{1+\kappa}, \overline{\theta}]$.

The following graphs show, for three different values of δ ($\delta = 0, 0.5$ and 1), the selectivity functions for a uniform distribution and for a cost $\kappa = 0.5$. The mapping $\phi(\cdot)$ is in solid lines, and the mapping $\gamma(\cdot)$ in dotted lines.

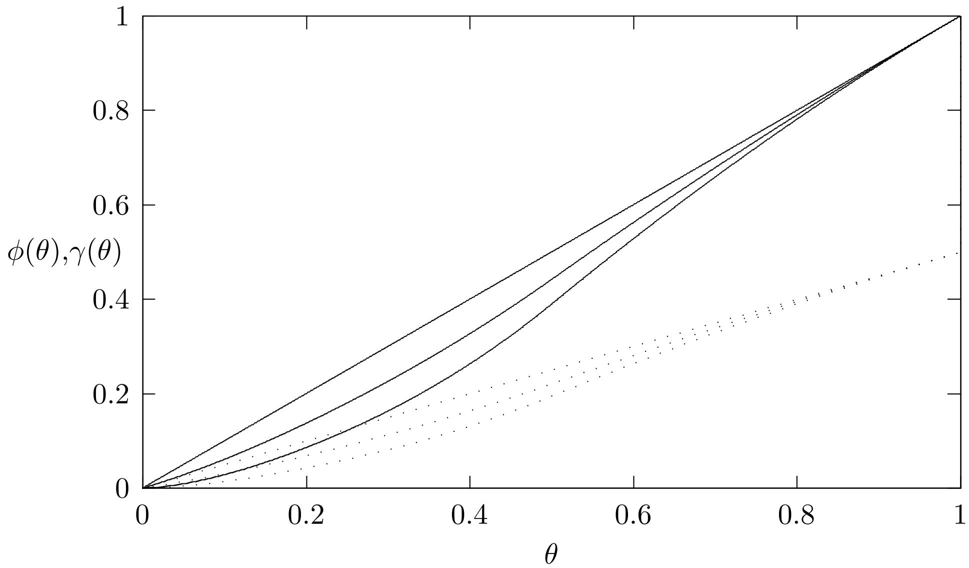


Figure 6: Optimal selectivity function (proportional recovery cost)

Figure 5 illustrates the difference between fixed and proportional recovery costs. It also shows an interesting non-monotonicity in the gap between the marginal type of the old and young agent, $\theta - \phi(\theta)$ with proportional recovery costs. For low values of θ, the cost of recovering objects becomes negligible, and hence the planner is willing to give objects to young agents more often, and the selectivity function is close to the first-best assignment rule, $\phi(\theta) = \theta$. Similarly, for high values of θ, as the probability of drawing an agent of type higher than θ becomes low, the gap $\theta - \phi(\theta)$ becomes small. The higher values of the gap $\theta - \phi(\theta)$ are obtained in the intermediate range, when the value of the object is neither too high nor too low.

4.2 Private information about agents' types

We now suppose that the types of the agents are not observable, and characterize revelation mechanisms which implement the optimal assignment rule. The structure of the model, where agents enter sequentially and the good may not be reassigned every period, allows for different timings in the revelation mechanisms. We will consider both a mechanism where agents pay transfers at the time they enter society, whether the good is reassigned or not (Mechanism I) and a mechanism where agents pay transfers at the time that the good is assigned (Mechanism II).

4.2.1 Mechanism I.

At any time t, the young agent is asked to reveal her type θ^y and pays a transfer $m_t(\theta^y, \theta^o)$ which may depend on the type of the old agent at time t. At any time $t \in T_1$, the mechanism designer chooses the probabilities with which the object is assigned to the young and old agents, $p_t^y(\theta^y, \theta^o)$ and $p_t^o(\theta^y, \theta^o)$.

Given this mechanism structure, the utility of an agent of type θ entering at time $t \in T_1$ and announcing $\hat{\theta}$ when the old agent is of type θ^o is given by

$$U_1(\theta, \hat{\theta}, \theta^o) = \theta(1+\delta)p_t^y(\hat{\theta}, \theta^o) + \delta(1 - p_t^y(\hat{\theta}, \theta^o))\theta E_{\theta^y} p_{t+1}^o(\theta^y, \hat{\theta}) - m_t(\hat{\theta}, \theta^o).$$

The utility of an agent of type θ entering at a time $t \in T_0$ and announcing $\hat{\theta}$ is independent of the type of the old agent at period t, and given by

$$U_0(\theta, \hat{\theta}) = \delta\theta E_{\theta^y} p_{t+1}^o(\theta^y, \hat{\theta}) - m_t(\hat{\theta}).$$

We consider the implementation of the optimal assignment rule, given by $p^y(\theta^y, \theta^o) = 1$ if $\theta^o \leq \phi(\theta^y)$ and $p^o(\theta^y, \theta^o) = 1$ if $\theta^o \geq \phi(\theta^y)$. The following proposition characterizes the transfer rules which implement the efficient assignment.

Proposition 5 *The efficient assignment can be implemented by a transfer rule m_t in mechanism 1 if and only if, for t in T_1,*

$$m_t(\theta, \theta^o) = \begin{cases} M_t \text{ if } \theta^o \leq \phi(\theta), \\ \delta\theta F(\phi^{-1}(\theta)) - \delta \int_{\underline{\theta}}^{\theta} F(\phi^{-1}(t))dt - N_t \text{ if } \theta^o \geq \phi(\theta) \end{cases}$$

where $N_t \geq 0$ and $M_t + N_t \leq (1+\delta)\phi^{-1}(\theta^o) - \delta \int_{\underline{\theta}}^{\phi^{-1}(\theta^o)} F(\phi^{-1}(t))dt$.

And, for $t \in T_0$,

$$m_t(\theta) = \delta\theta F(\phi^{-1}(\theta)) - \delta \int_{\underline{\theta}}^{\theta} F(\phi^{-1}(t))dt - N_t,$$

for $N_t \geq 0$.

Proposition 5 shows that the efficient assignment can be implemented by a stationary transfer rule, which only depends on whether the object is assigned at period t or not. For $\theta \geq \phi^{-1}(\theta^o)$,the efficient rule assigns the good to the young agent irrespective of the value θ. Hence, the transfer must be independent of θ.[7] For lower values of θ, (and in periods where the good is not reassigned), the agent receives the

[7]This is of course a variant of the classical result on the implementation of efficient auction rules and efficient public decisions using Vickrey-Clarke-Groves mechanisms.

good if and only if the young agent drawn next period has a type lower than $\phi^{-1}(\theta)$. Hence, the assignment probability of the good to the agent is strictly increasing in θ, implying that the transfer is strictly increasing in the announcement θ.

If we suppose furthermore that the planner incurs a cost of public funds, and wants to minimize her deficit, then $N_t = 0$ and the fixed payment paid by high type agents when the good is assigned is given by:

$$M = (1+\delta)\phi^{-1}(\theta^o) - \delta \int_{\underline{\theta}}^{\phi^{-1}(\theta^o)} F(\phi^{-1}(t))dt.$$

4.2.2 Mechanism II.

We now suppose that transfers are paid at the time that the good is assigned. Two situations may arise. If $t-1 \in T_1$, then the type of the old agent was revealed in the previous period. In that case, the planner will only set a transfer for the young agent, and the transfer may depend on the type of the old agent, $m_t(\theta^y, \theta^o)$. If, on the other hand, $t-1 \in T_0$, then the planner must make both agents reveal their types, and she will choose two transfer rules $m_t^y(\theta^y, \theta^o)$ and $m_t^o(\theta^y, \theta^o)$ for the young and old agent, respectively.

The utility of the young agent at a period t following a period $t-1$ in T_1 when his type is θ and he announces $\hat{\theta}$ is given by:

$$U_1(\theta, \hat{\theta}, \theta^o) = \theta(1+\delta)p_t^y(\hat{\theta}, \theta^o) + \delta(1 - p_t^y(\hat{\theta}, \theta^o))\theta E_{\theta^y} p_{t+1}^o(\theta^y, \hat{\theta}) - m_t(\hat{\theta}, \theta^o).$$

as in the case of young agents at periods $t \in T_1$ in mechanism I. The expected utility of a young and old agent when the previous period was in T_0 is given by:

$$\begin{aligned} EU_0^y &= \theta(1+\delta)F(\phi(\hat{\theta})) + \delta\theta(1 - F(\phi(\hat{\theta}))F(\phi^{-1}(\hat{\theta})) - \int_{\underline{\theta}}^{\overline{\theta}} m_t^y(\theta, \theta^o) f(\theta^o) d\theta^o, \\ EU_0^o &= \theta F(\phi^{-1}(\hat{\theta})) - \int_{\underline{\theta}}^{\overline{\theta}} m_t^o(\theta^y, \theta) f(\theta^y) d\theta^y, \end{aligned}$$

leading to the following characterization of the optimal revelation mechanisms:

Proposition 6 *The efficient assignment can be implemented by a transfer rule m_t in mechanism II if and only if, for any period t such that $t-1 \in T_1$,*

$$m_t(\theta, \theta^o) = \begin{cases} M_t \text{ if } \theta^o \leq \phi(\theta), \\ \delta\theta F(\phi^{-1}(\theta)) - \delta \int_{\underline{\theta}}^{\theta} F(\phi^{-1}(t))dt - N_t \text{ if } \theta^o \geq \phi(\theta) \end{cases}$$

where $N_t \geq 0$ *and* $M_t + N_t \leq (1+\delta)\phi^{-1}(\theta^o) - \delta \int_{\underline{\theta}}^{\phi^{-1}(\theta^o)} F(\phi^{-1}(t))dt$.

And for any period such that $t-1 \in T_0$,

$$\begin{aligned}
\int_{\underline{\theta}}^{\overline{\theta}} m_t^y(\theta,\theta^o) f(\theta^o) d\theta^o \quad &= \quad \theta(1+\delta)F(\phi(\theta)) + \delta\theta(1-F(\phi(\theta))F(\phi^{-1}(\theta)) \\
&+ \quad \int_{\underline{\theta}}^{\theta} (1+\delta)F(\phi(t)) + \delta(1-F(\phi(t))F(\phi^{-1}(t))dt \\
&-N_t, \\
\int_{\underline{\theta}}^{\overline{\theta}} m_t^o(\theta^y,\theta) f(\theta^y) d\theta^y \quad &= \quad \delta\theta F(\phi^{-1}(\theta)) - \delta \int_{\underline{\theta}}^{\theta} F(\phi^{-1}(t))dt \\
&- \quad N_t
\end{aligned}$$

where $N_t \geq 0$.

The efficient revelation mechanism in Model II differs from the mechanism in Model I because of difference in the information of the agents. In Model II, when both types are unknown, the planner uses a Bayesian mechanism to simultaneously obtain information about the types of the old and young agent. In that Bayesian mechanism, the expected utilities of the agents are strictly increasing in their types, so that the expected transfer rules are also strictly increasing in agents' types. By contrast, when the type of the old agent is known and the planner only needs to extract information about the type of the young agent, as in the efficient mechanism in Model I, the planner uses a flat transfer rule for high type agents.

4.3 Investment for type improvement

In this last extension, we allow agents to invest in order to improve their types. We consider the following timing. At the beginning of period t, before the type of the young agent is drawn, the old agent has access to a technology which increases his type from θ to θ' at a cost $c(\theta'-\theta)$. We assume that young agents cannot improve their types. [8]We let $c(\cdot)$ be a strictly increasing, convex function satisfying $c(0)=0$ and $c(\overline{\Delta})=1$ for some $\overline{\Delta}>0$. We distinguish between the investment of an old agent who does not hold the good and of an old agent who holds the good and denote the agents' types after investment as ζ and η in these two respective situations.

We characterize the planner's optimal choices when she has access to three controls: the probability $p^y(\theta^y,\theta)$, of assigning the good to the young agent, and the target type values of the old agent, $\zeta(\theta^o)$ and $\eta(\theta^o)$, in periods where the good is not

[8]Letting the agents invest before the good is assigned at their young age would greatly complicate the analysis without yielding additional insights.

assigned and in periods where the good is assigned respectively. We compute the expected values of the social planner before the type of the young agent is drawn in the cases where the old agent does not hold the good and holds the good as:

$$\begin{aligned} EV(\theta^o) = \max_{\zeta} \quad & -c(\zeta - \theta^o) + \int_{\underline{\theta}}^{\overline{\theta}} \max_{p^y} (p^y(\theta^y + \delta EW(\theta^y) \\ & +(1-p^y)(\zeta + \delta EV(\theta^y))f(\theta^y)d\theta^y, \end{aligned}$$

$$EW(\theta^o) = \max_{\eta} \quad \eta - c(\eta - \theta^o) + \int_{\underline{\theta}}^{\overline{\theta}} \delta EV(\theta^y) f(\theta^y) d\theta^y.$$

By a standard argument, we observe that the optimal assignment policy at any state (θ^y, θ) is to assign the good to the old agent whenever $\theta \geq \theta^y + \delta(EW(\theta^y) - EV(\theta^y))$. Furthermore, the optimal investment choices are determined by the first order conditions: $c'(\eta - \theta^o) = 1$ and $c'(\zeta - \theta^o) = \int_{\underline{\theta}}^{\overline{\theta}} p^y(\theta^y, \zeta)$. In particular, this shows that $\eta \geq \zeta$, so that an old agent who holds the good (and is sure to keep it next period) will invest more than an old agent who does not hold the good (and may not hold it next period). Furthermore, the optimal investment strategy of an old agent who holds the good is simply given by: $\eta = \theta + \overline{\Delta}$ for all θ.

When the planner faces an old and young agent of type $\overline{\theta}$, she is indifferent between giving the object to the old or young agent, as she will be collecting $\overline{\theta}$ at period t and $\overline{\theta} + \overline{\Delta} - c(\overline{\Delta})$ at period $t+1$ in both cases. This observation shows that, as in the baseline case, we obtain a simple boundary condition on the selectivity function $\phi(\overline{\theta}) = \overline{\theta}$.

Finally, we differentiate EV and EW and use the envelope Theorem to obtain:

$$\begin{aligned} \frac{\partial EV}{\partial \theta} = c'(\zeta - \theta^o) = \int_{\underline{\theta}}^{\overline{\theta}} p^y(\theta^y, \zeta), \\ \frac{\partial EW}{\partial \theta} = c'(\eta - \theta^o) = 1 \end{aligned}$$

and characterize the optimal assignment and investment strategies in the following Proposition:

Proposition 7 *When the agents can invest to improve their types, the optimal assignment and investment strategies, are characterized by a system of functional differential equations:*

$$\begin{aligned} \phi'(\theta) &= 1 + \delta[1 - F(\phi^{-1}(\zeta(\theta)))], \\ c'(\zeta(\theta) - \theta) &= F(\phi^{-1}(\zeta(\theta))), \\ c'(\eta(\theta) - \theta) &= 1 \end{aligned}$$

with the initial conditions $\eta(\overline{\theta}) = \zeta(\overline{\theta}) = \overline{\theta} + \overline{\Delta}$, $\phi(\overline{\theta}) = \overline{\theta}$.

Proposition 7 shows that the optimal assignment and investment strategies of the social planner are interdependent, and result from the simultaneous resolution of functional equations determining the selectivity function ϕ and the investment strategy ζ. For agents who already hold the object, the optimal investment policy is simply given by $\eta = \theta + \overline{\Delta}$. If $\overline{\Delta} < \overline{\theta} - \underline{\theta}$, an agent of type $\underline{\theta}$ will never invest to the point where he obtains the good with probability one. In that case, agents of low type, who may not obtain the good next period, invest less than agents who currently hold the good or agents of higher type who are guaranteed to hold the good next period.

Figure 7 illustrates how the optimal investment strategy ζ depends on an agent's type. The graph was obtained for a uniform distribution $F(\theta) = \theta$ and a quadratic cost function, $c(\Delta\theta) = \Delta\theta^2$. The three lines correspond to three values of the discount factor, $\delta = 0, 0.5, 1$, where the higher line corresponds to the lower discount factor. With this specification, $\overline{\Delta} = 1$, and we observe that there is a value $\overline{\theta} \sim 0.5$ such that all agents invest in order to improve their types to $\theta^o + 1$ when $\theta^o \geq \overline{\theta}$. For lower values of θ^o, agents are not guaranteed to hold the good next period, and the optimal investment strategy of the social planner is to invest less, so that $\zeta(\theta^o) < \theta^o + 1$. In particular, we observe that there exists a point of non-differentiability of the investment strategy at $\overline{\theta}$, with a left-hand derivative smaller than the right-hand derivative at this point.

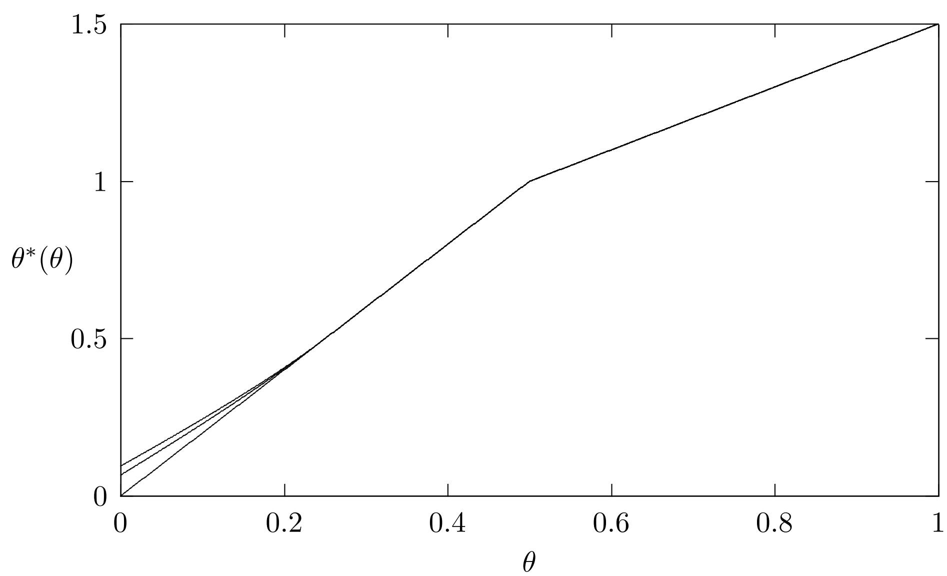


Figure 7: Optimal investment strategy

5 Conclusion

This paper analyzes the assignment of durable objects to successive generations of agents who live for two periods. Typical examples of durable objects are positions in organizations, offices, dormitory rooms or durable equipment like rental cars or large scale scientific equipment. We characterize the optimal assignment by a selectivity function which satisfies an iterative functional differential equation. We show that more patient social planners are more selective, as are social planners facing distributions of types with higher probabilities for higher types. We also investigate optimal assignment rules when agents can return the object at a cost, when the value of the object is private information, and when agents can invest to improve the match. In all these cases, we characterize the optimal assignment policy as a variation on the baseline selectivity function.

We are aware of several shortcomings of our analysis. First, because we limit attention to agents with a two-period lifetime, the trade-off between age and value reduces to a dichotomic choice between old and young agents. Increasing the agents' lifetime would allow us to obtain a finer characterization of the trade-off, but at the expense of a considerable increase the complexity of the model. Second, we suppose that agents enter and exit society through a simple deterministic process. In reality, agents enter and exit society through a stochastic birth and death process (possibly with memory, as older agents are more likely to die). Analyzing a model of assignment of durable goods with stochastic entry and exit is a challenging task on which we hope to make some progress in the future. Finally, we have limited our study of revelation mechanisms to efficient mechanisms. In some applications, like rental cars or hotel rooms, the planner may want to extract information about agents' values in order to maximize revenues rather than social surplus. The study of revenue-maximizing mechanisms for durable goods is a high priority on our research agenda.

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7 Appendix

Proof of Lemma 1: Let $h(\theta) = \int_{\underline{\theta}}^{\overline{\theta}} V(t,\theta)f(t)dt$ denote the expected value of the problem when the old agent is of type θ. Let $H = \int_{\underline{\theta}}^{\overline{\theta}} h(\theta)f(\theta)d\theta$ denote the expected value of h. We will show that for any $\theta < \theta'$, $h(\theta) - h(\theta') < \theta - \theta'$.
By definition of ϕ, the good will be assigned to the old agent of type θ at period $t+1$ if and only if the type of the young agent drawn next period is below $\phi(\theta)$. Hence

$$h(\theta) \quad = \quad \int_{t|\phi(t)\geq\theta} [t(1+\delta) + \delta^2 H]f(t)dt + \int_{t|\phi(t)\leq\theta} [\theta + \delta h(t)]f(t)dt.$$

so that

$$\begin{aligned}
h(\theta') - h(\theta) &= \int_{t|\theta\leq\phi(t)\leq\theta'} [t(1+\delta) + \delta^2 H - \delta h(t)]f(t)dt + \int_{t|\phi(t)\leq\theta'} \theta' f(t)dt - \int_{t|\phi(t)\leq\theta} \theta f(t)dt, \\
&= -\int_{t|\theta\leq\phi(t)\leq\theta'} \phi(t)f(t)dt + \int_{t|\phi(t)\leq\theta'} \theta' f(t)dt - \int_{t|\phi(t)\leq\theta} \theta f(t)dt, \\
&\leq -\int_{t|\theta\leq\phi(t)\leq\theta'} \theta f(t)dt + \int_{t|\phi(t)\leq\theta'} \theta' f(t)dt - \int_{t|\phi(t)\leq\theta} \theta f(t)dt, \\
&= (\theta' - \theta) \int_{t|\phi(t)\leq\theta'} f(t)dt, \\
&\leq \theta' - \theta.
\end{aligned}$$

For any $\theta < \theta'$, as $h(\theta) - h(\theta') < \theta - \theta'$, $\phi(\theta) < \phi(\theta')$, showing that ϕ is increasing.

To prove the second statement, notice that the planner could always choose to assign the good to the old agent at period $t+1$, so that

$$h(\theta) \geq \theta + \delta H.$$

The inequality shows that $\phi(\theta) \leq \theta$. This inequality will be strict for all $\theta < \overline{\theta}$, as there is always a positive probability to draw a young agent in $(\theta, \overline{\theta}]$ which would give a higher surplus to the planner. As $h(\overline{\theta}) = \overline{\theta} + \delta H$, we obtain $\phi(\overline{\theta}) = \overline{\theta}$, as claimed in the statement of the Lemma.

Proof of Theorem 1: First note that if $\phi(\theta)$ is differentiable, $h(\theta) \equiv \int_{\underline{\theta}}^{\overline{\theta}} V(t,\theta)f(t)dt$ is also differentiable, and we can compute:

$$h'(\theta) = \int_{t\leq\phi^{-1}(\theta)} f(t)dt = F(\phi^{-1}(\theta)).$$

Differentiating $\phi(\cdot)$, we obtain:

$$\phi'(\theta) = 1 + \delta - \delta F(\phi^{-1}(\theta)).$$

Finally, we prove the existence and uniqueness of a differentiable function satisfying equation (2) in an open ball around $\theta \in (\underline{\theta}, \overline{\theta})$. To this end, we will consider a change of variable, and define the inverse function of ϕ as $\chi = \phi^{-1}$.

The differential equation (2) now becomes:

$$\chi'(\theta) = \frac{1}{1 + \delta(1 - F(\chi \circ \chi)(\theta))}.$$

Finally, we prove the existence and uniqueness of a differentiable function satisfying equation (2) in an open ball around $\theta \in (\underline{\theta}, \overline{\theta})$. To this end, let $\chi(\theta) \equiv \hat{\theta}$ be fixed, and consider the set $\mathcal{C}$ of bounded real-valued continuous functions over the open ball $B_\epsilon(\theta)$ satisfying the following two conditions: $\chi(\theta) = \hat{\theta}$ and $|\chi(t_1) - \chi(t_2)| < |t_1 - t_2|$ for all $(t_1, t_2) \in B_\epsilon(\theta)$. Clearly, $\mathcal{C}$ is a closed subset of the complete metric space of bounded real-valued continuous functions, equipped with the sup norm, $||\chi|| = \sup_{t \in B_\epsilon(\theta)} |\chi(t)|$.

Next, consider the following operator T on $\mathcal{C}$,

$$T\chi(t) = \hat{\theta} + \int_\theta^t \frac{1}{1 + \delta - \delta F[(\chi \circ \chi)(z)]} dz.$$

We first show that T maps $\mathcal{C}$ onto itself. Clearly, $T\chi(\theta) = \hat{\theta}$. Furthermore, for any $t_1, t_2 \in B_\epsilon(\theta)$,

$$\begin{aligned} |T\chi(t_1) - T\chi(t_2)| &= |\int_{t_1}^{t_2} \frac{1}{1 + \delta - \delta F[(\chi \circ \chi)(z)]} dz|, \\ &< |t_1 - t_2|, \end{aligned}$$

where the last inequality is obtained because $\frac{1}{1+\delta-\delta F[(\chi \circ \chi)(z)]} < 1$. Next we show that T is a contracting operator. For any t,

$$\begin{aligned} |T\chi(t) - T\chi'(t)| &= |\int_\theta^t \frac{1}{1 + \delta - \delta F[(\chi \circ \chi)(z)]} - \frac{1}{1 + \delta - \delta F[(\chi' \circ \chi')(z)]} dz, \\ &= |\int_\theta^t \frac{\delta(F(\chi' \circ \chi')(z) - F(\chi \circ \chi)(z))dz}{(1 + \delta - \delta F[(\chi \circ \chi)(z)])(1 + \delta - \delta F[(\chi' \circ \chi')(z)])}| \\ &< |\int_\theta^t \delta(F(\chi' \circ \chi')(z) - F(\chi \circ \chi)(z))dz| \\ &\leq \delta|\int_\theta^t (\chi' \circ \chi')(z) - (\chi \circ \chi)(z))dz| \end{aligned}$$

where the first inequality is due to the fact that $\frac{1}{1+\delta-\delta F[(\chi\circ\chi)(z)]} < 1$ and the second to the fact that the distribution function F is differentiable and hence Lipschitz continuous. Now,

$$\begin{aligned} |(\chi'\circ\chi')(z)-(\chi\circ\chi)(z))| &\leq |(\chi'\circ\chi')(z)-(\chi\circ\chi')(z))|+|(\chi\circ\chi')(z)-(\chi\circ\chi)(z)|, \\ &\leq |\chi'(\chi'(z))-\chi(\chi'(z))|+|\chi'(z)-\chi(z)| \end{aligned}$$

where the last inequality is due to the fact that χ is Lipschitz continuous. Integrating over z, we find that

$$\begin{aligned} |T\chi(t)-T\chi'(t)| &< 2(t-\theta)\sup_{\tau\in B_\epsilon(\theta)}|\chi'(\tau)-\chi(\tau)|, \\ &\leq 2\epsilon||\chi-\chi'||, \end{aligned}$$

so that T is a contracting operator. By Banach's fixed point theorem, the operator admits a unique fixed point χ which satisfies $\chi(\theta)=\hat{\theta}$ and $\chi'(\theta)=\frac{1}{1+\delta-\delta F[(\chi\circ\chi)(\theta)]}$.

Proof of Proposition 1: We recall that:

$$\frac{\partial\phi(\theta,\delta)}{\partial\theta}=1+\delta-\delta F(\phi^{-1}(\theta)).$$

Differentiating again with respect to δ:

$$\frac{\partial^2\phi(\theta,\delta)}{\partial\theta\partial\delta}=1-F(\phi^{-1}(\theta,\delta))-\delta f(\phi^{-1}(\theta,\delta))\frac{\partial\phi^{-1}(\theta,\delta)}{\partial\delta}. \tag{7}$$

Notice that $\frac{\partial^2\phi(\overline{\theta},\delta)}{\partial\theta\partial\delta}=1-F(\phi^{-1}(\overline{\theta},\delta)=\frac{\partial\phi^{-1}(\overline{\theta},\delta)}{\partial\delta}=0$. We differentiate once more with respect to θ:

$$\begin{aligned} \frac{\partial^3\phi(\theta,\delta)}{\partial\theta^2\partial\delta} &= -f(\phi^{-1}(\theta,\delta)\frac{\partial\phi^{-1}(\theta,\delta)}{\partial\theta} \\ &- \delta f'(\phi^{-1}(\theta,\delta))\frac{\partial\phi^{-1}(\theta,\delta)}{\partial\theta}\frac{\partial\phi^{-1}(\theta,\delta)}{\partial\delta} \\ &- \delta f(\phi^{-1}(\theta,\delta))\frac{\partial^2\phi^{-1}(\theta,\delta)}{\partial\theta\partial\delta}. \end{aligned}$$

Evaluating this derivative at $\overline{\theta}$ and using the fact that $\frac{\partial\phi^{-1}(\overline{\theta},\delta)}{\partial\delta}=\frac{\partial^2\phi^{-1}(\overline{\theta},\delta)}{\partial\theta\partial\delta}=0$ and $\frac{\partial\phi^{-1}(\theta,\delta)}{\partial\theta}>0$, we conclude that:

$$\frac{\partial^3\phi(\overline{\theta},\delta)}{\partial\theta^2\partial\delta}<0.$$

By continuity, there exists an open interval $(\overline{\theta}-\epsilon,\overline{\theta})$ such that $\frac{\partial^3\phi(\theta,\delta)}{\partial\theta^2\partial\delta}<0$ for all $\theta\in(\overline{\theta}-\epsilon,\overline{\theta})$. But as $\frac{\partial^2\phi(\overline{\theta},\delta)}{\partial\theta\partial\delta}=0$, this implies that $\frac{\partial^2\phi(\theta,\delta)}{\partial\theta\partial\delta}>0$ for all $\theta\in(\overline{\theta}-\epsilon,\overline{\theta})$ and, as $\frac{\partial\phi(\overline{\theta},\delta)}{\partial\delta}=0$, that $\frac{\partial\phi(\theta,\delta)}{\partial\delta}<0$ for all $\theta\in(\overline{\theta}-\epsilon,\overline{\theta})$.
We want to show that $\frac{\partial\phi(\theta,\delta)}{\partial\delta}<0$ for all δ and all $\theta\in[\underline{\theta},\overline{\theta})$. To this end, we will prove the (stronger) statement that $\frac{\partial^2\phi(\theta,\delta)}{\partial\theta\partial\delta}>0$ for all δ and all θ. Fix some $\delta>0$ and suppose by contradiction that there exists $\tilde{\theta}$ such that $\frac{\partial^2\phi(\theta,\delta)}{\partial\theta\partial\delta}>0$ for all $\theta\in(\tilde{\theta},\overline{\theta})$ but $\frac{\partial^2\phi(\tilde{\theta},\delta)}{\partial\theta\partial\delta}=0$. Let

$$\psi(\theta,\delta)\equiv\frac{1}{f(\phi^{-1}(\theta,\delta)}\frac{\partial^2\phi(\theta,\delta)}{\partial\theta\partial\delta}=\lambda(\phi^{-1}(\theta,\delta))-\delta\frac{\partial\phi^{-1}(\theta,\delta)}{\partial\delta},$$

Because $f(\phi^{-1}(\theta,\delta))>0$ for all $\theta\in(\underline{\theta},\overline{\theta}]$, we have:

$$\frac{\partial^2\phi(\theta,\delta)}{\partial\theta\partial\delta}>0 \quad\Leftrightarrow\quad \psi(\theta)>0,$$
$$\frac{\partial^2\phi(\theta,\delta)}{\partial\theta\partial\delta}=0 \quad\Leftrightarrow\quad \psi(\theta)=0.$$

In particular, we know that $\psi(\tilde{\theta})=\psi(\overline{\theta})=0$ and $\psi(\tilde{\theta})>0$ for all $\theta\in(\tilde{\theta},\overline{\theta})$. Next, note that

$$\frac{\partial\phi^{-1}(\theta,\delta)}{\partial\theta}=\frac{1}{\frac{\partial\phi(\phi^{-1}(\theta,\delta),\delta)}{\partial\theta}}$$

so that

$$\frac{\partial^2\phi^{-1}(\theta,\delta)}{\partial\theta\partial\delta}=-\frac{\frac{\partial^2\phi(\phi^{-1}(\theta,\delta),\delta)}{\partial\theta\partial\delta}+\frac{\partial^2\phi(\phi^{-1}(\theta,\delta),\delta)}{\partial\theta^2}\frac{\partial\phi^{-1}(\theta\delta)}{\partial\delta}}{(\frac{\partial\phi(\phi^{-1}(\theta,\delta)}{\partial\theta})^2}.$$

Notice that, for all $\theta\in[\underline{\theta},\overline{\theta})$, $\frac{\partial^2\phi(\theta,\delta)}{\partial\theta^2}<0$ and $\frac{\partial\phi^{-1}(\tilde{\theta},\delta)}{\partial\delta}=\frac{\lambda(\phi^{-1}(\tilde{\theta},\delta))}{\delta}>0$. Hence,

$$-\frac{\partial^2\phi^{-1}(\theta,\delta)}{\partial\theta\partial\delta}<(\frac{\partial\phi^{-1}(\theta,\delta)}{\partial\theta})^2\frac{\partial^2\phi(\phi^{-1}(\theta,\delta),\delta)}{\partial\theta\partial\delta}.$$

Next, note that as $\phi^{-1}(\theta,\delta)>\theta$, $\phi^{-1}(\theta,\delta)>\tilde{\theta}$ and hence $\frac{\partial\phi^{-1}(\phi^{-1}(\theta,\delta),\delta)}{\partial\delta}>0$. This implies that

$$\frac{\partial^2\phi(\phi^{-1}(\theta,\delta),\delta)}{\partial\theta\partial\delta}<1-F(\phi^{-1}(\phi^{-1}(\theta,\delta),\delta)<1-F(\phi^{-1}(\theta,\delta)), \tag{8}$$

where the first inequality stems from equation (7) and the second from the fact that F is increasing. Furthermore, using equation (7), we compute:

$$\delta f(\phi^{-1}(\tilde{\theta},\delta))\frac{\partial \phi^{-1}(\tilde{\theta},\delta)}{\partial \delta} = 1 - F(\phi^{-1}(\tilde{\theta},\delta)). \tag{9}$$

Using equations (8) and (9), we obtain:

$$\begin{aligned}
\frac{\partial^3 \phi(\tilde{\theta},\delta)}{\partial \theta^2 \partial \delta} &< -f(\phi^{-1}(\tilde{\theta},\delta))\frac{\partial \phi^{-1}(\tilde{\theta},\delta)}{\partial \theta} \\
&- \frac{f'(\phi^{-1}(\tilde{\theta},\delta))(1-F(\phi^{-1}(\tilde{\theta},\delta)))}{f(\phi^{-1}(\tilde{\theta},\delta))}\frac{\partial \phi^{-1}(\tilde{\theta},\delta)}{\partial \theta} \\
&+ \delta f(\phi^{-1}(\tilde{\theta},\delta))(1-F(\phi^{-1}(\tilde{\theta},\delta)))(\frac{\partial \phi^{-1}(\tilde{\theta},\delta)}{\partial \theta})^2.
\end{aligned}$$

Next, using the fact that $0<\delta<1$, $0<\frac{\partial \phi^{-1}(\theta,\delta)}{\partial \theta}<1$ and $0<f(\theta)$, we have:

$$\frac{\partial^3 \phi(\tilde{\theta},\delta)}{\partial \theta^2 \partial \delta} < \frac{1}{f(\phi^{-1}(\tilde{\theta},\delta)}(\lambda'(\phi^{-1}(\tilde{\theta},\delta)) + f^2(\phi^{-1}(\tilde{\theta},\delta))(1-F(\phi^{-1}(\tilde{\theta},\delta)))). \tag{10}$$

By a similar argument, we compute:

$$\frac{\partial \psi(\tilde{\theta},\delta)}{\partial \theta} < \lambda'(\phi^{-1}(\tilde{(\theta)},\delta)) + 1 - F(\phi^{-1}(\tilde{\theta},\delta)). \tag{11}$$

Using Assumption 1, we have that $\lambda'(\theta)+(1-F(\theta))\min\{1,f(\theta)^2\}<0$ for all θ. This implies that either $\frac{\partial^3 \phi(\tilde{\theta},\delta)}{\partial \theta^2 \partial \delta}<0$ or $\frac{\partial \psi(\tilde{\theta},\delta)}{\partial \theta}<0$. But hence, we either contradict the fact that $\frac{\partial^2 \phi(\tilde{\theta},\delta)}{\partial \theta \partial \delta}=0$ or the fact that $\psi(\tilde{\theta},\delta)=0$. This argument thus shows that, for all $\theta\in[\theta^*(\delta),\overline{\theta})$, $\frac{\partial^2 \phi(\theta,\delta)}{\partial \theta \partial \delta}>0$ and hence $\frac{\partial \phi(\theta,\delta)}{\partial \delta}<0$, concluding the proof.

Proof of Proposition 2: We compute

$$\frac{\partial^2 \phi(\theta,\alpha)}{\partial \theta \partial \alpha} = -\delta\frac{\partial F(\phi^{-1}(\theta,\alpha)}{\partial \alpha} - \delta f(\phi^{-1}(\theta,\alpha))\frac{\partial \phi^{-1}(\theta,\alpha)}{\partial \alpha}. \tag{12}$$

Note that $\frac{\partial^2 \phi(\overline{\theta},\alpha)}{\partial \theta \partial \alpha} = \frac{\partial F(\phi^{-1}(\overline{\theta},\alpha))}{\partial \alpha} = \frac{\partial \phi^{-1}(\overline{\theta},\alpha)}{\partial \alpha} = 0$. Differentiating once more with respect to θ:

$$\begin{aligned}
\frac{\partial^3 \phi(\theta,\alpha)}{\partial \theta^2 \partial \alpha} &= -\delta\frac{\partial f(\phi^{-1}(\theta,\alpha))}{\partial \alpha}\frac{\partial \phi^{-1}(\theta,\alpha)}{\partial \theta} \\
&\quad -\delta f'(\phi^{-1}(\theta,\alpha))\frac{\partial \phi^{-1}(\theta,\alpha)}{\partial \theta}\frac{\partial \phi^{-1}(\theta,\alpha)}{\partial \alpha} \\
&\quad -\delta f(\phi^{-1}(\theta,\alpha))\frac{\partial^2 \phi^{-1}(\theta,\alpha)}{\partial \theta \partial \alpha}.
\end{aligned}$$

We note that $\frac{\partial F(\overline{\theta},\alpha)}{\partial \alpha} = 0$ and $\frac{\partial F(\theta,\alpha)}{\partial \alpha} < 0$ for $\theta \in (\underline{\theta}, \overline{\theta})$. This implies that $\frac{\partial f(\phi^{-1}(\overline{\theta},\alpha))}{\partial \alpha} > 0$. Hence, we observe that

$$\frac{\partial^3 \phi(\overline{\theta}, \alpha)}{\partial \theta^2 \partial \alpha} < 0.$$

In turn, this implies that there exists an open interval $(\overline{\theta} - \epsilon, \overline{\theta})$ such that, for all $\theta \in (\overline{\theta} - \epsilon, \overline{\theta})$, $\frac{\partial^3 \phi(\overline{\theta},\alpha)}{\partial \theta^2 \partial \alpha} < 0$, $\frac{\partial^2 \phi(\theta,\alpha)}{\partial \phi \partial \alpha} > 0$ and $\frac{\partial \phi(\theta,\alpha)}{\partial \alpha} < 0$.
We aim to show that for all α and $\theta \in [\underline{\theta}, \overline{\theta})$, $\frac{\partial \phi(\theta,\alpha)}{\partial \alpha} < 0$. To this end, we will prove the stronger statement that $\frac{\partial^2 \phi(\theta,\alpha)}{\partial \phi \partial \alpha} > 0$ for all α, θ. Suppose by contradiction, that there exists α and $\tilde{\theta}$ such that such that $\frac{\partial^2 \phi(\theta,\alpha)}{\partial \phi \partial \alpha} > 0$ for all $\theta > \tilde{\theta}$ and $\frac{\partial^2 \phi(\tilde{\theta},\alpha)}{\partial \phi \partial \alpha} = 0$. Define the mapping

$$\xi(\theta, \alpha) \equiv \frac{1}{\delta F(\phi^{-1}(\theta, \alpha))} \frac{\partial^2 \phi(\theta, \alpha)}{\partial \phi \partial \alpha} = -\mu(\phi^{-1}(\theta, \alpha)) - \nu(\phi^{-1}(\theta, \alpha)) \frac{\partial \phi^{-1}(\theta, \alpha)}{\partial \alpha}.$$

Because $\delta, F(\phi^{-1}(\theta, \alpha)) > 0$, we know that

$$\begin{aligned} \xi(\theta, \alpha) = 0 \quad &\Leftrightarrow \quad \frac{\partial^2 \phi(\theta, \alpha)}{\partial \phi \partial \alpha} = 0, \\ \xi(\theta, \alpha) > 0 \quad &\Leftrightarrow \quad \frac{\partial^2 \phi(\theta, \alpha)}{\partial \phi \partial \alpha} > 0. \end{aligned}$$

In particular, this shows that $\xi(\theta, \alpha) > 0$ for all $\theta \in (\tilde{\theta}, \overline{\theta})$ and $\xi(\tilde{\theta}, \alpha) = 0$. Now compute

$$\begin{aligned} \frac{\partial \xi(\theta, \alpha)}{\partial \theta} \quad = \quad & -\mu'(\phi^{-1}(\theta, \alpha)) \frac{\partial \phi^{-1}(\theta, \alpha)}{\partial \theta} \\ & -\nu'(\phi^{-1}(\theta, \alpha)) \frac{\partial \phi^{-1}(\theta, \alpha)}{\partial \theta} \frac{\partial \phi^{-1}(\theta, \alpha)}{\partial \alpha} \\ & -\nu(\phi^{-1}(\theta, \alpha)) \frac{\partial^2 \phi^{-1}(\theta, \alpha)}{\partial \theta \partial \alpha}. \end{aligned}$$

Now, note that, by definition of $\tilde{\theta}$,

$$\frac{\partial \phi^{-1}(\tilde{\theta}, \alpha)}{\partial \alpha} = \frac{\frac{\partial F(\phi^{-1}(\tilde{\theta},\alpha))}{\partial \alpha}}{f(\phi^{-1}(\tilde{\theta}, \alpha))} = \frac{\mu(\phi^{-1}(\theta, \alpha))}{\nu(\phi^{-1}(\theta, \alpha))}.$$

Furthermore,

$$\frac{\partial^2\phi^{-1}(\theta,\alpha)}{\partial\theta\partial\alpha} = -(\frac{\partial\phi^{-1}(\theta,\alpha)}{\partial\theta})^2(\frac{\partial^2\phi(\phi^{-1}(\theta,\alpha),\alpha)}{\partial\theta\partial\alpha} - \frac{\partial^2\phi(\phi^{-1}(\theta,\alpha),\alpha)}{\partial\theta^2}\frac{\partial\phi^{-1}(\theta,\alpha)}{\partial\alpha}).$$

Because $\frac{\partial^2\phi(\phi^{-1}(\tilde{\theta},\alpha),\alpha)}{\partial\theta^2} < 0$ and $\frac{\partial\phi^{-1}(\tilde{\theta},\alpha)}{\partial\alpha} > 0$, we have:

$$-\frac{\partial^2\phi^{-1}(\tilde{\theta},\alpha)}{\partial\theta\partial\alpha} < (\frac{\partial\phi^{-1}(\tilde{\theta},\alpha)}{\partial\theta})^2\frac{\partial^2\phi(\phi^{-1}(\tilde{\theta},\alpha),\alpha)}{\partial\theta\partial\alpha}. \tag{13}$$

Furthermore, because $\phi^{-1}(\tilde{\theta}) > \tilde{\theta}$,

$$\begin{aligned}\frac{\partial^2\phi(\phi^{-1}(\tilde{\theta},\alpha),\alpha)}{\partial\theta\partial\alpha} &< -\delta\frac{\partial F(\phi^{-1}(\phi^{-1}(\tilde{\theta},\alpha),\alpha))}{\partial\alpha},\\ &< -\frac{\partial F(\phi^{-1}(\phi^{-1}(\tilde{\theta},\alpha),\alpha))}{\partial\alpha}.\end{aligned}$$

Using the previous inequality and inequality (13),

$$\begin{aligned}\frac{\partial^2\phi(\phi^{-1}(\tilde{\theta},\alpha),\alpha)}{\partial\theta\partial\alpha} &< \frac{\frac{\partial^2\phi(\phi^{-1}(\tilde{\theta},\alpha),\alpha)}{\partial\theta\partial\alpha}}{F((\phi^{-1}(\phi^{-1}(\tilde{\theta},\alpha),\alpha))},\\ &< -\frac{\frac{\partial F(\phi^{-1}(\phi^{-1}(\tilde{\theta},\alpha),\alpha))}{\partial\alpha}}{F((\phi^{-1}(\phi^{-1}(\tilde{\theta},\alpha),\alpha))},\\ &= -\mu(\phi^{-1}(\phi^{-1}(\tilde{\theta},\alpha),\alpha)),\\ &< -\mu(\phi^{-1}(\tilde{\theta},\alpha)).\end{aligned}$$

where the first inequality comes from the fact that $F((\phi^{-1}(\phi^{-1}(\tilde{\theta},\alpha),\alpha))) < 1$ and the last from the fact that $\mu(\theta)$ is increasing.

Combining these inequalities, we get:

$$\frac{\partial\xi(\tilde{\theta},\alpha)}{\partial\theta} < -\mu'(\phi^{-1}(\tilde{\theta},\alpha)) - \nu'(\phi^{-1}(\tilde{\theta},\alpha))\frac{\mu(\phi^{-1}(\tilde{\theta},\alpha))}{\nu(\phi^{-1}(\tilde{\theta},\alpha))} - \nu(\phi^{-1}(\tilde{\theta},\alpha))\mu(\phi^{-1}(\tilde{\theta},\alpha)).$$

Using assumption 2, we conclude that $\frac{\partial\xi(\tilde{\theta},\alpha)}{\partial\theta} < 0$, contradicting the fact that $\xi(\tilde{\theta},\alpha) = 0$ and $\xi(\theta,\alpha) > 0$ for all $\theta > \tilde{\theta}$.

Proof of Proposition 3: Omitted.

Proof of Proposition 4: Omitted.

Proof of Proposition 5: Because the efficient assignment is stationary, we will consider stationary transfer rules, which only depend on whether the good is reassigned or not at period t and let m_1 and m_0 denote the transfer rules for $t \in T_1$ and $t \in T_0$. Consider a period $t \in T_1$. Given the efficient assignment, the utility of an agent is given by:

$$U_1(\theta, \hat{\theta}, \theta^o) = \begin{cases} \theta(1+\delta) - m_1(\hat{\theta}, \theta^o) \text{ if } \hat{\theta} \geq \phi^{-1}(\theta^o), \\ \delta\theta F(\phi^{-1}(\hat{\theta})) - m_1(\hat{\theta}, \theta^o) \text{ if } \hat{\theta} \leq \phi^{-1}(\theta^o) \end{cases}$$

Hence, it is clear that, for $\theta \geq \phi^{-1}(\theta^o)$, the transfer rule must be constant, and $m_1(\theta, \theta^o) = M$. Using standard arguments, for $\theta \leq \phi^{-1}(\theta^o)$,

$$\frac{\partial U_1}{\partial \theta} = \delta F(\phi^{-1}(\theta)),$$

so that

$$U = U(\underline{\theta}) + \delta \int_{\underline{\theta}}^{\theta} F(\phi^{-1}(t))dt.$$

Letting $N = U(\underline{\theta})$, we obtain:

$$m_1 = \delta\theta F(\phi^{-1}(\theta)) - \int_{\underline{\theta}}^{\theta} F(\phi^{-1}(t))dt - N.$$

Finally, we need to guarantee that an agent of type $\theta = \phi^{-1}(\theta^o)$ does not have an incentive to announce $\theta < \phi^{-1}(\theta^o)$. This will be true as long as:

$$\phi^{-1}(\theta^o)(1+\delta) - M \geq \delta \int_{\underline{\theta}}^{\phi^{-1}(\theta^o)} F(\phi^{-1}(t))dt + N.$$

For $t \in T_0$, we have

$$U_0(\theta, \hat{\theta}) = \delta\theta F(\phi^{-1}(\hat{\theta})) - m_0(\hat{\theta}).$$

We simply compute the transfer m_0 as:

$$m_0 = \delta\theta F(\phi^{-1}(\theta)) - \int_{\underline{\theta}}^{\theta} F(\phi^{-1}(t))dt - N.$$

Proof of Proposition 6: For periods t following a period $t-1 \in T_1$, when the type of the old agent is known, the mechanism is the same as that of Proposition 5 and the same argument applies.

For periods such that neither type is known, we simply note that

$$\frac{\partial EU_0^y}{\partial \theta} = (1+\delta)F(\phi(\theta)) + \delta(1 - F(\phi(\theta))F(\phi^{-1}(\theta)),$$

and that

$$\begin{aligned} EU_0^y(\theta) &= \int_{\underline{\theta}}^{\theta} \frac{\partial EU_0^y}{\partial \theta} dt + EU_0^y(\underline{\theta}), \\ &= \int_{\underline{\theta}}^{\theta} (1+\delta)F(\phi(t)) + \delta(1 - F(\phi(t))F(\phi^{-1}(t))dt - M, \end{aligned}$$

to get the desired transfer, and use the same argument to compute the expected transfer of the old agent, noticing that

$$\frac{\partial EU_0^o}{\partial \theta} = F(\phi^{-1}(\theta)).$$

Proof of Proposition 7: Because the planner can always choose to assign the good to the old agent, we necessarily have $EV(\theta^o) \geq EW(\theta^o)$. This implies that $\phi(\theta^y) < \theta^y$, so that the planners always prefers to assign the good to the old agent when $\theta^o = \theta^y$. We conclude that no agent will ever have an incentive to invest beyond $\overline{\theta} + \overline{\Delta}$ and any agent of type $\overline{\theta}$ chooses to invest up to $\overline{\theta} + \overline{\Delta}$ whether he holds the good or not, namely $\eta(\overline{\theta}) = \zeta(\overline{\theta}) = \overline{\theta} + \overline{\Delta}$. This in turn implies that $\phi(\overline{\theta}) = \overline{\theta}$ as the social planner is indifferent between giving the object to a young or old agent of type $\overline{\theta}$. Assuming that the optimal policy is differentiable, we compute $\frac{\partial EV}{\partial \theta}$ and $\frac{\partial EW}{\partial \theta}$ and observe that $\phi'(\theta) > 0$, showing that the optimal policy of the planner is an increasing selectivity function. The remainder of the proof follows from standard arguments.